



Introduction

The theory of linear anelasticity and viscoelasticity explains very satisfactorily the phenomenon encountered in the real geomaterials: a higher material attenuation goes hand in hand with a steeper increase of the velocity of seismic waves with increasing frequency.

There is a causal relationship between the stress applied to the rock and the resulting strain or vice versa. If a linear system is causal, then Kramers – Kronig formulae connect the real and imaginary part of its complex frequency response – in our case, the storage and loss shear modulus:

$$G'(\omega) = \lim_{\omega \rightarrow \infty} G'(\omega) - \frac{2}{\pi} \text{PV} \int_0^{\infty} \frac{\omega' G''(\omega') - \omega G''(\omega)}{\omega'^2 - \omega^2} d\omega'$$

$$G''(\omega) = \frac{2\omega}{\pi} \text{PV} \int_0^{\infty} \frac{G'(\omega') - G'(\omega)}{\omega'^2 - \omega^2} d\omega'$$

where PV denotes the Cauchy principal value of the integral.

For minimum phase systems, Bayard – Bode relationships hold for the logarithm of magnitude and phase – in our case, the logarithm of magnitude and loss angle of the complex viscoelastic shear modulus:

$$\alpha(\omega) = \lim_{\omega \rightarrow 0} \alpha(\omega) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d(\phi(u)/\exp(u))}{du} \ln \coth \frac{|u|}{2} du$$

$$\phi(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(u)}{du} \ln \coth \frac{|u|}{2} du$$

where $u = \ln(\omega'/\omega)$.

The Kramers – Kronig or Bayard – Bode relationships provide a direct test of the consistency of the complex viscoelastic modulus.

For practical reasons, it is much more convenient to use an indirect test based on relaxation time spectra. A suitably shaped, non-negative and continuous logarithmic relaxation time spectrum H can describe any type of physically admissible linear viscoelastic or anelastic behaviour in shear:

$$G'(\omega) = G_R + \int_{-\infty}^{\infty} \frac{\omega^2 \tau^2 H(\ln \tau)}{1 + \omega^2 \tau^2} d(\ln \tau)$$

$$G''(\omega) = \int_{-\infty}^{\infty} \frac{\omega \tau H(\ln \tau)}{1 + \omega^2 \tau^2} d(\ln \tau)$$

where the relaxed shear modulus $G_R = \lim_{\omega \rightarrow 0} G'(\omega)$.

The relaxation time spectrum H can be computed by inversion of either storage or loss modulus. If the complex viscoelastic modulus is consistent according to Kramers – Kronig relationships, the two spectra are identical. The less consistent the modulus is, the greater is their difference.

We extended the concept of the relaxation time spectrum to the relaxation time spectra of the magnitude and loss angle of the viscoelastic shear modulus:

$$\phi(\Omega) = \int_{-\infty}^{\infty} \arctan \frac{\exp(\Omega + \Theta) h(\Theta)}{1 + \exp(2\Omega + 2\Theta)} d\Theta$$

$$\ln |G^*(\Omega)| = \ln G_R + \frac{1}{2} \int_{-\infty}^{\infty} \ln \left(1 + \frac{h^2}{2} + \frac{h}{2} \sqrt{4 + h^2} \right) d\Theta + \int_{-\infty}^{\infty} \arctan \frac{h \tanh(\Omega + \Theta)}{\sqrt{4 + h^2}} d\Theta$$

where $\Theta = \ln(\tau)$, $\Omega = \ln(\omega)$.

In this case, the h-spectra are unscaled, whereas the H-spectra of storage and loss modulus are scaled with a scaling factor whose physical dimension is [Pa]. The table of inversion methods to test the viscoelastic consistency is herewith completed:

inversion	direct	via relaxation time spectrum
$G'(\omega) \leftrightarrow G''(\omega)$	Kramers – Kronig	standard $G'(\omega) \leftrightarrow H(\Theta) \leftrightarrow G''(\omega)$
$\ln G^*(\omega) \leftrightarrow \phi(\omega)$	Bayard – Bode	our solution $\ln G^*(\omega) \leftrightarrow h(\Theta) \leftrightarrow \phi(\omega)$

The frequency responses of seismic velocity and quality factor could be inverted comfortably to the unscaled magnitude and loss angle spectra respectively, as the unscaling during the velocity inversion effectively eliminates density (which is constant with respect to frequency) and the quality factor is the reciprocal of the tangent of the loss angle. See the section “Field experiment – a sketch of the solution” for more details.

Reference: Bednárík M., Kohút I., 2019: Relaxation time spectra of basaltic lavas between 500–1150°C reveal patterns of Kramers – Kronig inconsistency of the complex viscoelastic shear modulus. *Contrib. Geophys. Geod.*, **49**, 2 (in press).

Laboratory experiment

The cylindrical specimens of three different types of basaltic lavas kept in a protective atmosphere at atmospheric pressure were subject to harmonic torque of various frequencies at stepwise increasing temperatures. The sinusoids of the shear stress and shear strain were compared and the magnitude and loss angle of the shear modulus computed.

We reprocessed the data of the original publication James et al. (2004). After the inversion of storage and loss modulus or magnitude and loss angle to the respective relaxation time spectra, we used the ratio of the latter couple of spectra

$$h_{|G'|}(\Theta) / h_{\phi}(\Theta)$$

as an indicator of the viscoelastic shear modulus inconsistency.

With temperature increase from 500°C to 1100°C, the viscoelastic consistency of all examined lava samples improved to the ideal linear viscoelasticity. Nevertheless, each sample has a specific fingerprint. In the Etna 1992 samples, there is a not very convincing evidence for fairly good consistency at ~500°C, then there is a loss of consistency between ~600°C and ~800°C, especially dramatic in Etna 1992 top sample at 786°C, where the ratio of the relaxation time spectra approaches 20. This inconsistency level remains a mystery and a very strong motivation for a further search for extreme inconsistencies and physical conditions for their existence.

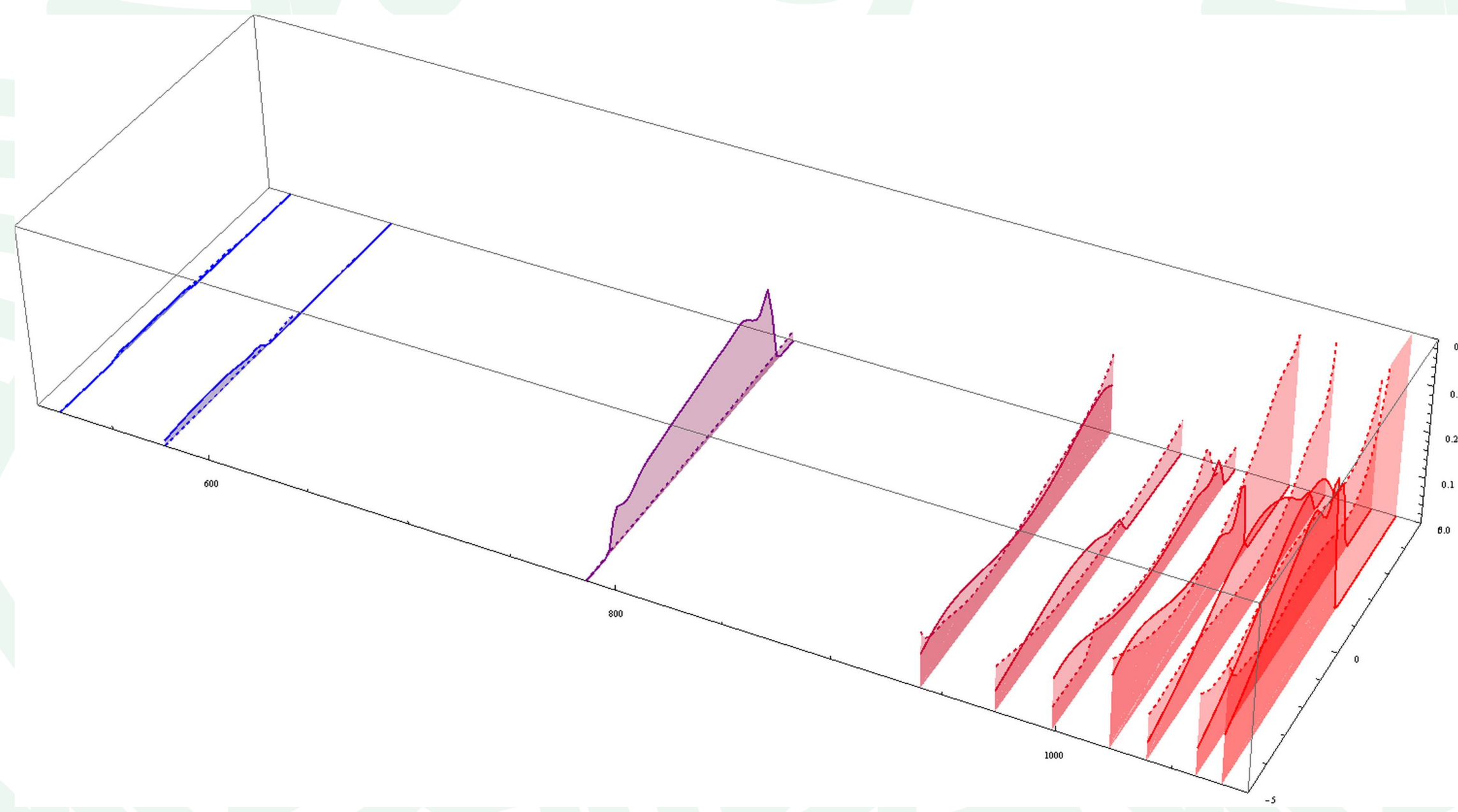


Fig.1 Development of the magnitude-borne (solid outline) and loss angle-borne (dashed outline) relaxation time spectra with increasing temperature in the Etna lava sample taken from the top of the lava flow

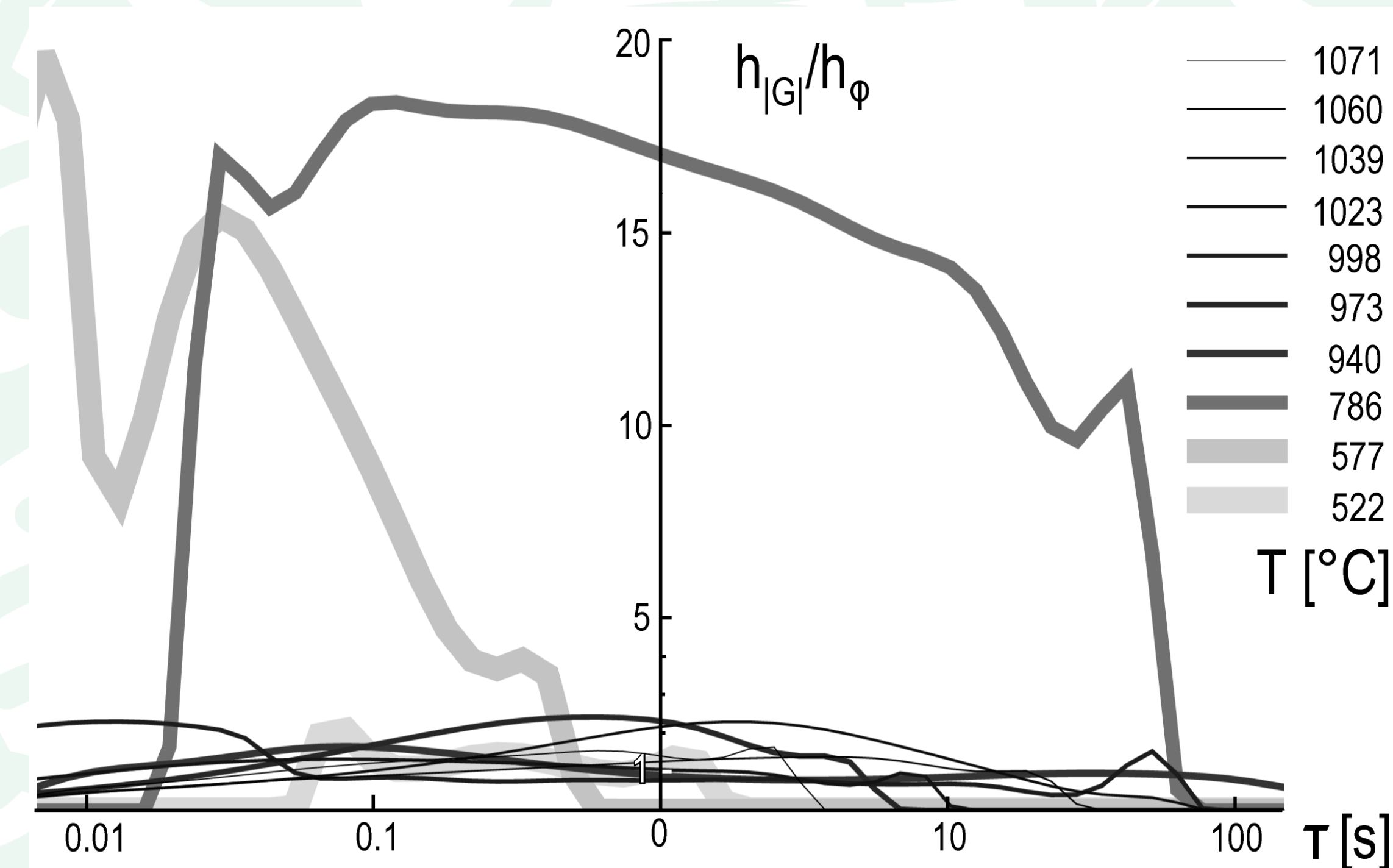


Fig.2 Ratios of magnitude-borne and loss angle-borne relaxation time spectra for different temperatures in the Etna lava sample taken from the top of the lava flow

Reference: James M. R., Bagdassarov N., Müller K., Pinkerton H., 2004: Viscoelastic behaviour of basaltic lavas. *J. Volcanol. Geotherm. Res.*, **132**, 2-3, 99–113.

Field experiment – a sketch of the solution

Let us consider once again the viscoelastic shear modulus and the possibilities to test its consistency by seismic data. The velocity of shear waves depends on the shear modulus as

$$v_s(\omega) = \sqrt{G'(\omega) / \rho}$$

As the density does not depend on frequency, the first derivative of the logarithm of velocity reads

$$\frac{d \ln(v_s(\omega))}{d \omega} = \frac{1}{2} \frac{d \ln(G'(\omega))}{d \omega} = \frac{1}{2\omega} \frac{d \ln(G'(\Omega))}{d \Omega}$$

For a reasonably high Q , $G'(\omega) \approx |G^*(\omega)|$ and the above equation can be combined with

$$\frac{d \ln |G^*(\Omega)|}{d \Omega} = \int_{-\infty}^{\infty} \frac{h \sqrt{4 + h^2}}{2 + h^2 + 2 \cosh 2(\Omega + \Theta)} d\Theta$$

to provide the input data for the inversion of shear wave velocity frequency response to the unscaled relaxation time spectrum $h_{|G'|}(\Theta)$.

Another theoretical possibility is to use the difference of group and phase velocity of the shear waves:

$$\frac{dG'(\Omega)}{d\Omega} = 2G'(\Omega) \left(1 - \frac{v_s}{v_{sg}} \right)$$

As for the quality factor, the intrinsic material attenuation is just one component of the overall apparent attenuation, the other one is the attenuation due to scattering (after Boxberger et al., 2017):

$$Q_{app}^{-1}(\Omega) \approx Q_i^{-1}(\Omega) + Q_{sc}^{-1}(\Omega)$$

It is likely that the unscaled relaxation time spectra extracted from quality factor frequency responses will be biased. This should not be a big setback, if we are looking for spatial and temporal variations of the ratio of the spectra. Another source of their bias could be the incompatible methods for velocity and quality factor computation.

Once the frequency dependence of the quality factor for the shear waves is available, its inversion to the respective relaxation time spectrum is straightforward:

$$\phi(\Omega) = \arctan Q_s^{-1}(\Omega) = \arctan(G''(\Omega)/G'(\Omega))$$

$$\phi(\Omega) = \int_{-\infty}^{\infty} \arctan \frac{\exp(\Omega + \Theta) h(\Theta)}{1 + \exp(2\Omega + 2\Theta)} d\Theta$$

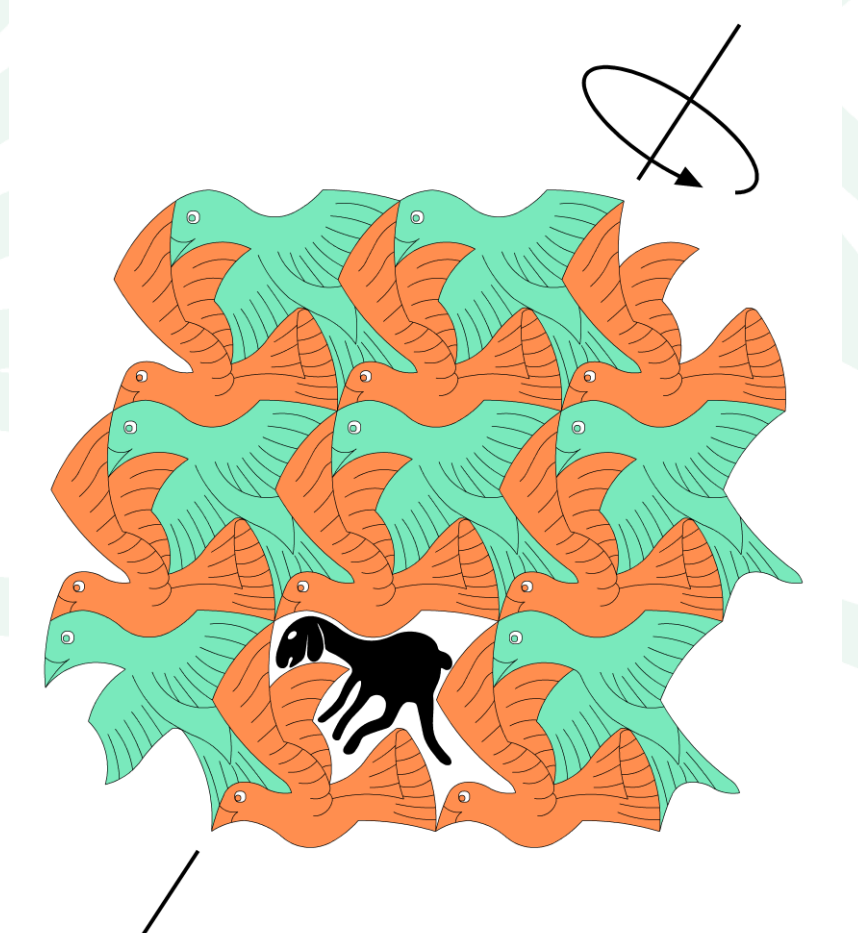
If the $Q_s^{-1}(\Omega)$ is nearly constant, so is the $h_{\phi}(\Theta)$. Instead of exact inversion, the relationship between the constants can be used:

$$h_{\phi} = 2\phi / \pi$$

Methodically, we are prepared for the inversion of velocity and quality factor frequency responses to the respective relaxation time spectra to assess the consistency of the viscoelastic shear modulus in the Earth's interior. We are well aware that the inversion of seismograms into the spatial distribution of frequency-dependent seismic velocities and quality factors poses a much more difficult problem. We hope that we inspired the specialists in seismic tomography to take the challenge. Maybe they will be rewarded by new perspectives of remote defectoscopy usable both at regional and very local scale (microstructural changes in rocks due to a nuclear explosion?). Until now, we do not know anything about the spatial and temporal extent and distribution of the viscoelastic inconsistencies. It is possible that their spatial scale is lesser than the available seismic wavelengths. Nevertheless, even such a local inconsistency with a large magnitude could raise the average inconsistency level in a broader area and thus become detectable. We will never know before you try.

Reference: Boxberger T., Pilz M. and Parolai S., 2017: Shear wave velocity versus quality factor: results from seismic noise recordings. *Geophys. J. Int.*, **210**, 660–670.

Would you like to express your opinion or suggestion? Feel free to write your message onto the margins or send me an email to Martin.Bednarik@savba.sk



Reference to the graphics: Escher M. C., 1949: Birds (wood engraving)